Laminar Flow Heat Transfer

E. B. CHRISTIANSEN, GORDON E. JENSEN, and FAN-SHENG TAO

University of Utah, Salt Lake City, Utah

Numerical solutions of the equations of motion and energy for the heating of non-Newtonian fluids in rectilinear, axisymmetric laminar flow in tubes of circular cross section are extended to the case of cooling at constant tube-wall temperature. The fluid density, heat capacity, and thermal conductivity are assumed constant, but the flow properties are represented by the temperature-dependent equation $\tau = m(\dot{S} \exp \Delta H^{\ddagger}/RT)^n$. The results are presented as plots of average N_{Nu} vs. N_{Gz}, with flow and temperature-effect indices as parameters. The mean deviation of experimental data for the cooling of aqueous CMC and carbopol dispersions from the numerical solutions is ±8%.

Many solutions of the equations of motion and energy for laminar flow in a tube of circular cross section and for a number of simplifying restrictions have been reported. These are reviewed and analyzed in references *8* and *20*. Most of these solutions (*4, 5, 13* to *15, 20, 28*) are restricted to fluids whose properties do not vary with temperature. Ordinarily the variation of the fluid thermal conductivity and heat capacity with temperature is small, and the variation of the fluid density with temperature is of major importance only in cases in which natural convection plays a significant role in heat transfer or if pressure gradients are very large. However, the restriction to temperature-independent flow properties is serious, since under most circumstances the dependence of fluid consistency on temperature is a major influence on the heat transfer process (*9, 10, 20*). Recently (*9, 10*) finite-difference numerical solutions were reported for the heating of fluids whose flow is represented by a temperature-dependent form of the Ostwald-deWaele power law equation (*9*):

$$\tau = m[\dot{S} \exp (\Delta H^{\ddagger}/RT)]^n \tag{1}$$

and a temperature-dependent Powell-Eyring equation (*10*):

$$\tau = \mu[\dot{S} \exp (\Delta H^{\ddagger}/RT)] + 1/B \sinh^{-1} \left[\frac{\dot{S} \exp (\Delta H^{\ddagger}/RT)}{A} \right] \tag{2}$$

in which m, n, $\Delta H^{\ddagger}$, A, B, and μ are considered independent of temperature.

These published solutions for temperature-dependent flow (*9, 10*) embody the following restrictions:

1. The flow is steady state.
2. The flow is rectilinear and axisymmetric.
3. The fluid thermal conductivity k, heat capacity C_p, and density ρ are independent of temperature and pressure.
4. Isothermal flow is fully developed at the entrance to the heat transfer section.
5. Thermal energy generation in the fluid by viscous dissipation or any reaction is negligible.
6. Axial diffusion of energy and momentum is negligible.
7. The fluid is heated at constant wall temperature.

Although Equation (1) is unrealistic at very high and very low shear rates (predicts infinite apparent viscosity at very low shear rates and zero apparent viscosity as the shear rates become infinite), Equation (1) has been shown (*2, 26*, p. 203) to represent the flow of many nonassociating Newtonian fluids ($n = 1$) and some non-Newtonian fluids over extended shear rate ranges (*9, 23*). Also for many engineering purposes it can be used to approximate the flow of many additional non-Newtonian fluids in applications where the important shear rate range is relatively narrow.

In the present application, the velocity profile in the region of relatively high shear rates near the tube wall is of major importance. Consequently if Equation (1) represents flow for these shear rates reasonably well, the results will be satisfactory for most purposes and will not be greatly affected by the failure of Equation (1) to represent flow near the tube center. On the other hand, Equation (2) predicts realistic viscosities in the limits of very high and very low shear rates and has been found to represent accurately the data for many pseudoplastic fluids (*10* to *12, 27*) and the data for some slurries which are ordinarily considered to be Bingham plastic (*11, 31*). However, there are many fluids which exhibit an intermediate power law shear rate–shear stress region, which extends over such a wide range that it cannot be represented by Equation (2) [see, for example, reference *23*]; and in these cases Equation (1) is advantageous. Also Equation (1) may be easier to apply than is Equation (2).

The usefulness of these solutions is not seriously limited by restrictions to steady state, rectilinear axisymmetric flow, since most practical processes of laminar flow in tubes with heat transfer occur under approximately these conditions. While the effects of the small variations in heat capacity and thermal conductivity with temperature are almost always negligible, the variation of density with temperature is often important since it generates free convection and may also affect the temperature distribution if pressure gradients are high. Free convection alters the velocity profile and in the case of horizontal tubes leads to nonrectilinear, nonaxisymmetric flow. However, if N_{Gz} is relatively large, and N_{Gr}/N_{Re}^2 (*1*) and $N_{Pr} N_{Gr} D/L$ are small, which is the case in a large proportion of the industrial applications, natural convection can be neglected. Furthermore, except at impractically low flow rates (*22*) the neglect of natural convection leads to conservative results. Viscous dissipation of energy becomes important at very high velocities or in the case of fluids with very high viscosities or low thermal conductivities or combinations of these (*31*) so that N_{Br} is large.

The neglect of axial diffusion may restrict the application to fluids for which $N_{Pr} > 0.5$ or to $N_{Pr}N_{Re} \gtrsim 40$ to 100 (*16, 29*). Additional research is required to fix these limits. If flow at the tube entrance is not fully developed, the results will be conservative and in some cases uneco-

Gordon Jensen is with the United Technology Center, Sunnyvale, California. Fan-Sheng Tao is with Torginol of America, Inc., Huntington Park, California.

nomically, so if the tube length is much less than about one half the flow entrance length, which has been reported (8, 18) as a function of n, the exponent of Equation (1). However, in most practical circumstances, entrance lengths are very short because of the relatively high consistencies and N_{Pr} of most non-Newtonian fluids of importance in the process industries. Consequently the solutions can be applied in many practical circumstances in which the heated section is not preceded by a calming section, as well as in cases when it is.

The presently reported research extends the solution for power law flow heating (9, 12) to the case of cooling in a tube at constant wall temperature.

NUMERICAL PROCEDURES AND RESULTS

With restrictions 1, 2, 3, and 6 the equation of motion reduces to

$$-\frac{\partial P}{\partial z} = \frac{1}{r}\frac{\partial (r\,\tau_{rz})}{\partial r} \qquad (3)$$

By taking $\partial P/\partial z$ independent of r, we can integrate Equation (3) at a given z to yield

$$-\frac{r}{2}\frac{dP}{dz} = \tau_{rz} \qquad (4)$$

Also

$$-\frac{R}{2}\frac{dP}{dz} = \tau_{Rz}$$

and

$$\tau_{rz} = \tau_{Rz}\frac{r}{R} = \tau_{Rz}\, r^+ \qquad (5)$$

The substitution of (1) into (5) yields

$$m\left[\frac{1}{R}\left(-\frac{du}{dr^+}\right)\exp\,(\Delta H^\ddagger/RT)\right]^n = r^+\ \tau_{Rz}$$

or

$$du = -R\left[\frac{\tau_{Rz}}{m}\right]^{\frac{1}{n}}\exp\,(-\Delta H^\ddagger/RT)\ (r^+)^{\frac{1}{n}}\,dr^+ \qquad (6)$$

Integrate between $u = 0$ to $u = u$ and $r^+ = 1$ to $r^+ = r^+$ at a given z and

$$u = R\left[\frac{\tau_{Rz}}{m}\right]^{\frac{1}{n}}\int_{r^+}^{1}(r^+)^{\frac{1}{n}}\exp\,(\Delta H^\ddagger/RT)\,dr^+$$

or

$$u = R\left[\frac{\tau_{Rz}}{m}\right]^{\frac{1}{n}} I_1 \qquad (7)$$

The mass flow rate in the tube is

$$w = 2\pi\rho\int_0^R r u dr = \text{constant}$$

and by (7)

$$w = 2\pi\rho R^3\int_0^1\left[\frac{\tau_{Rz}}{m}\right]^{\frac{1}{n}} I_1\ r^+ dr^+$$

or

$$w = 2\pi\rho R^3\left[\frac{\tau_{Rz}}{m}\right]^{\frac{1}{n}} I_2 \qquad (8)$$

By (7) and (8)

$$u = \frac{w}{2\pi R^2\rho}\frac{I_1}{I_2} \qquad (9)$$

With restrictions 1, 2, 3, 5, and 6, the energy equation reduces to

$$\frac{\partial T}{\partial z} = \frac{\alpha}{ur}\left[\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] = \frac{\alpha}{ur}\left(\frac{\partial T}{\partial r} + r\frac{\partial^2 T}{\partial r^2}\right)$$

and by (9)

$$\frac{\partial T^+}{\partial z^+} = 2\pi\frac{I_2}{I_1}\left(\frac{1}{r^+}\frac{\partial T^+}{\partial r^+} + \frac{\partial^2 T^+}{\partial r^{+2}}\right) \qquad (10)$$

Equation (10) was integrated by numerical methods (12, 17) for the boundary condition of constant tube wall temperature T_R to yield a solution

$$N_{Nu_a} = f\left(n, N_{Gz}, \Delta H^\ddagger/RT_R, \frac{T_R - T_i}{T_i}\right) \qquad (11)$$

In the analysis the dimensionless groups $\Delta H^\ddagger/RT_R$ and $(T_R - T_i)/T_i$ occur as a product which suggests the use of $\psi(H) = (\Delta H^\ddagger/RT_R)(T_R - T_i/T_i)$ as a parameter. The computed N_{Nu} for both heating and cooling are plotted as a function of N_{Gz} for n of 1.0, 0.5, 0.3, and 0.1 in Figures 1, 2, 3, and 4 respectively with $\psi(H)$ as a parameter. The value of $\psi(H)$ is positive for heating and negative for cooling. It was established that for values of $\psi(H)$ from 0.0 to $\pm$ 2.0 and +3 the computed results for an arbitrary value of $\Delta H^\ddagger/RT_R$ of 10 are within $\pm$ 2% of the functions plotted in Figures 1, 2, 3, and 4 for $\Delta H^\ddagger/RT_R$ of 5 to 20. However for $\psi(H)$ of −3 the $\Delta H^\ddagger/RT_R$ becomes a significant parameter, particularly at high N_{Gz}, and the computed solutions for $\Delta H^\ddagger/RT_R$ of 5 and 20 are included as broken lines in Figures 1 to 4. The upper limiting case represented by the function for $\psi(H) = \infty$ [rod flow (14)] is included for comparison. It should be pointed out, however, that at relatively low N_{Gz} and relatively high temperature gradients, free convection effects may generate N_{Nu} greater than those for rod flow.

In making the finite-difference numerical computations the tube flow was considered to consist of 100 concentric sleeves of equal radial thickness. Because of the small radial intervals employed, the negligible variation of the results with mesh size, and the rigor of the computational procedures used, it is believed that the computations are accurate to within 1%. The solutions have been shown (9) to be in excellent agreement with earlier isothermal [$\psi(H) = 0$] solutions for Newtonian and power law non-Newtonian fluids (13, 14, 19). Furthermore, a recent independent numerical solution (18), in which radial flow is accounted for, is in essentially exact agreement with the presently reported solution for $n = 1$ and $\psi(H) = 1$.

COMPARISON OF COMPUTED RESULTS WITH EXPERIMENTAL DATA

The solutions have been shown previously (9) to be in good agreement with experimental data for heating aqueous 3% by weight CMC (carboxymethylcellulose) and 0.75% by weight CPM (carboxypolymethylene) at constant tube-wall temperature. These data were for flow in 1-, 1½-, and 2-in. copper pipe for L/D ratios of 6 to 230, for temperature increases to 30°C., and for temperature potentials to 70°C. Also, the recent data of Oliver and Jenson (22) for heating aqueous 1.6% Polyox ($n = 0.3$, $\psi(H) = 0.7$) in 1½-in. tubes are in reasonable agreement with the computed functions. However, the data of Oliver and Jenson are at N_{Gz} below 130, where the effect of $\psi(H)$ is small, and consequently the comparison is not very discriminating.

Experimental data of this study for the cooling of aqueous 1.6% by weight CMC and 0.35% by weight CPM dispersions at constant tube-wall temperatures (30) are compared with computed functions in Figure 5. The deviations of the experimental data from the computed functions (average 8.5% for the CMC and 7.2% for the Car-

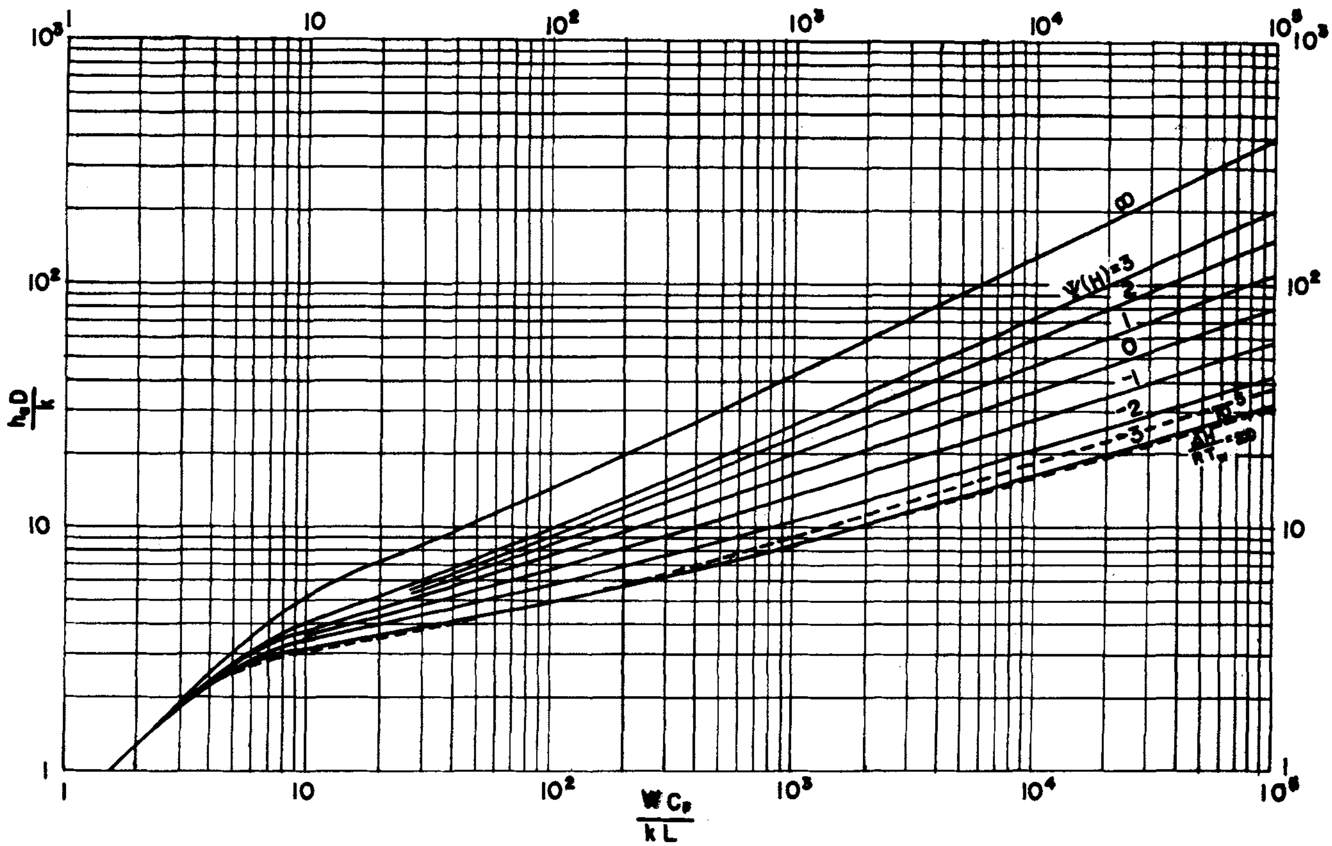

Fig. 1. Computed N_{Nu_a} plotted vs. N_{Gz} with $\psi(H)$ and $\Delta H^\ddagger/RT_R$ as parameters for Newtonian fluids ($n = 1$). The lines are for $\Delta H^\ddagger/RT_R = 10$. Dashed lines are included for $\Delta H^\ddagger/RT_R = 5$ and 20 when for these values the function differed more than 2% from that for $\Delta H^\ddagger/RT_R = 10$. Rod flow is represented by $\psi(H) = \infty$.

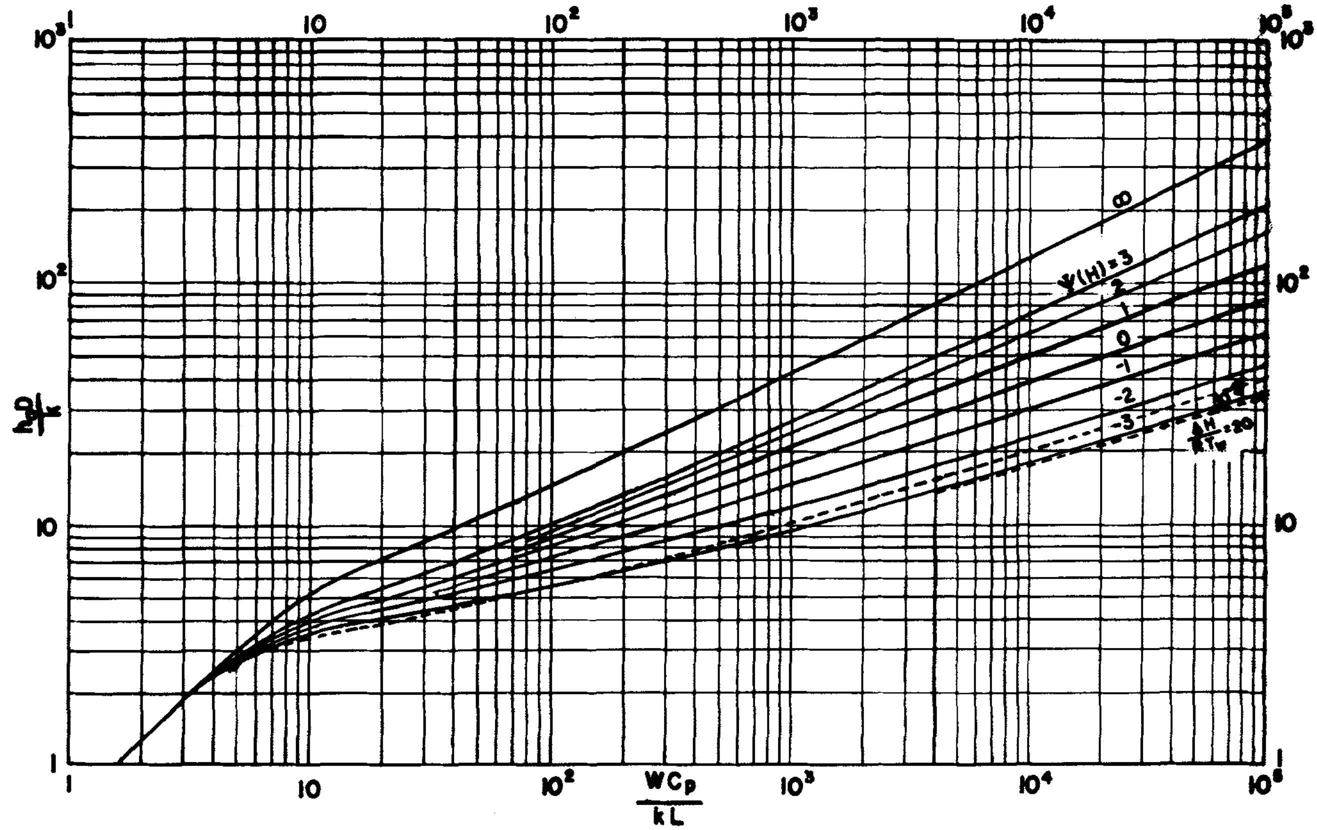

Fig. 2. Computed N_{Nu_a} plotted vs. N_{Gz} with $\psi(H)$ and $\Delta H^\ddagger/RT_R$ as parameters for $n = 0.5$. The solid lines are for $\Delta H^\ddagger/RT_R = 10$. Dashed lines are included for $\Delta H^\ddagger/RT_R = 5$ and 20 when for these values the function differed more than 2% from that for $\Delta H^\ddagger/RT_R = 10$.

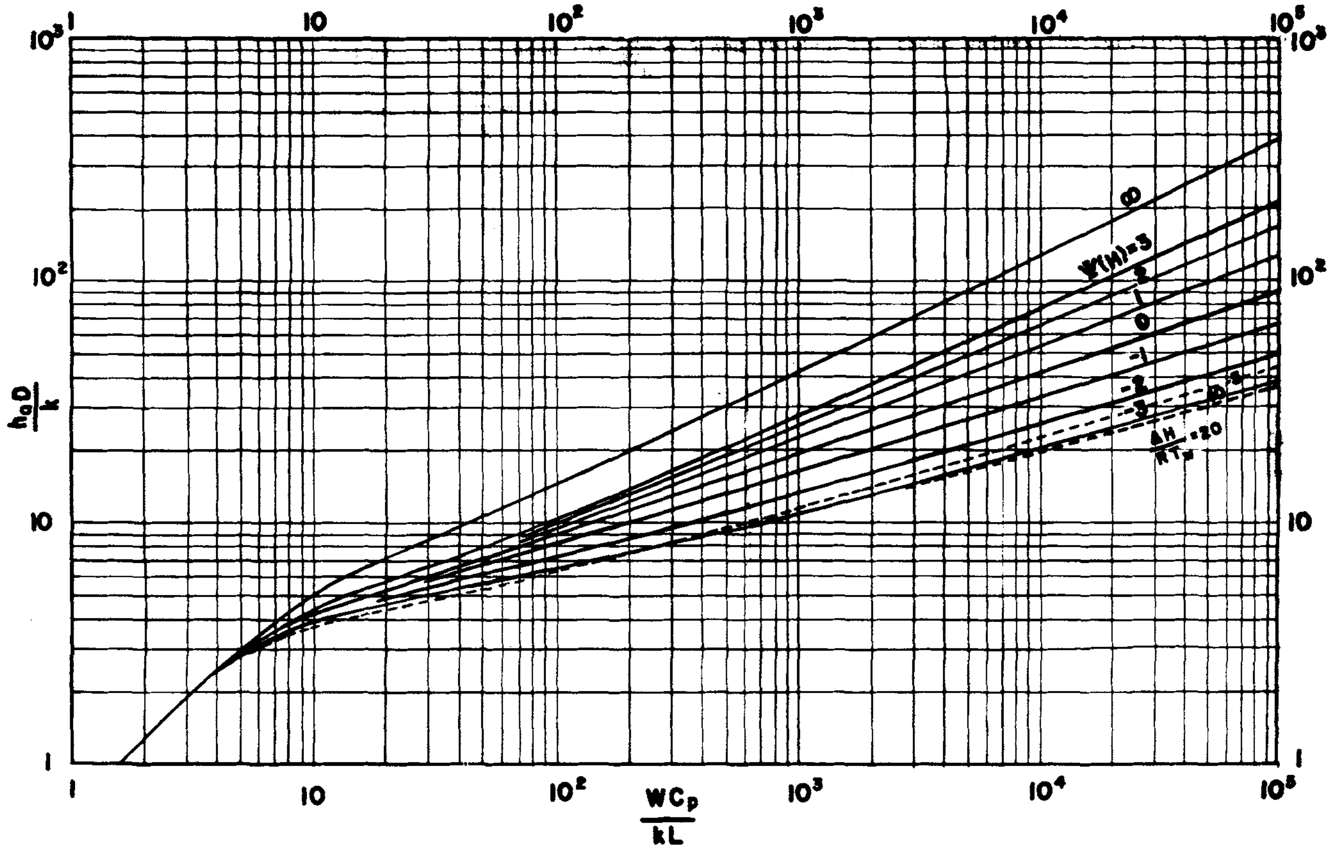

Fig. 3. Computed N_{Nu_a} plotted vs. N_{Gz} with $\psi(H)$ and $\Delta H^{\ddagger}/RT_R$ as parameters for $n = 0.3$. The solid lines are for $\Delta H^{\ddagger}/RT_R = 10$. Dashed lines are included for $\Delta H^{\ddagger}/RT_R = 5$ and 20 when for these values the function differed more than 2% from that for $\Delta H^{\ddagger}/RT_R = 10$.

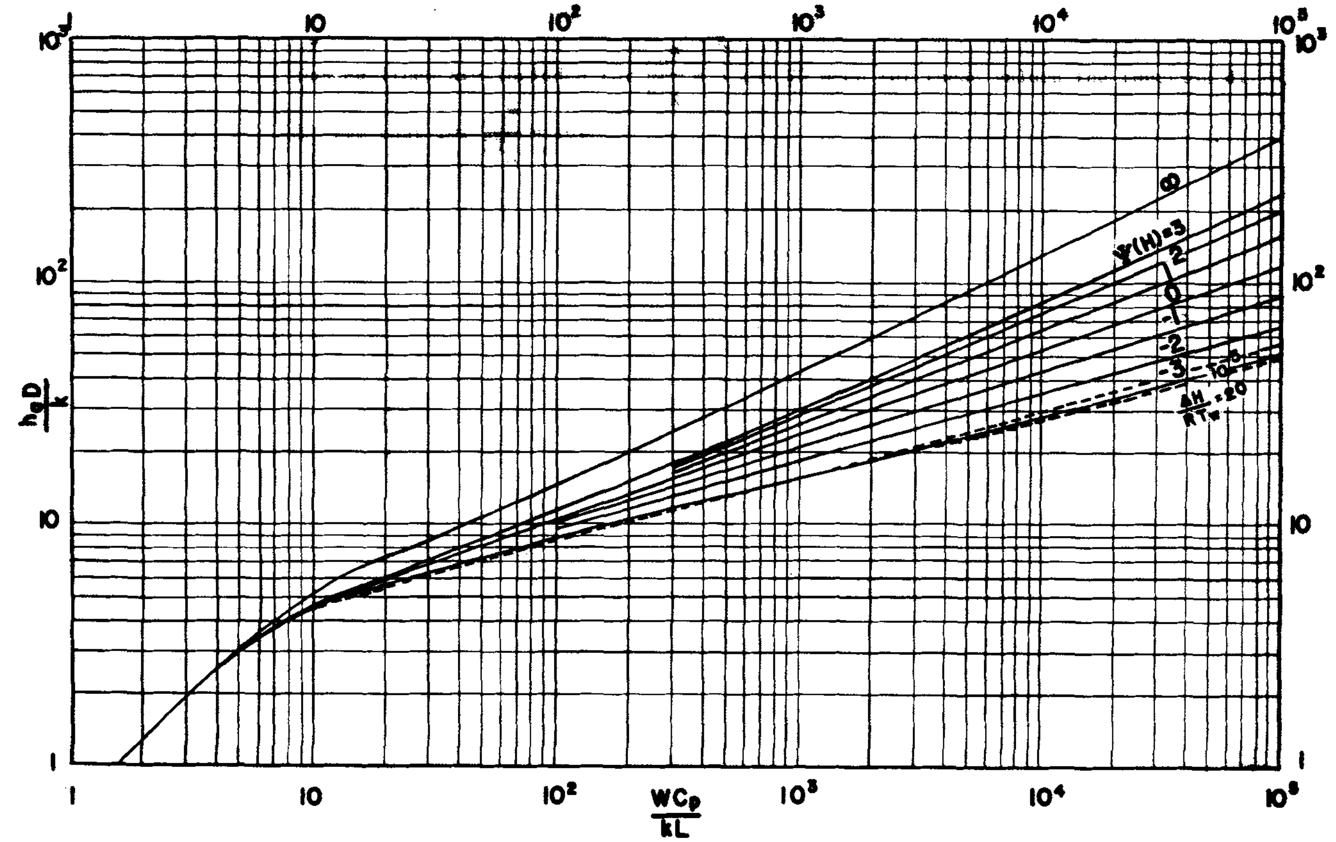

Fig. 4. Computed N_{Nu_a} plotted vs. N_{Gz} with $\psi(H)$ and $\Delta H^{\ddagger}/RT_R$ as parameters for $n = 0.1$. The solid lines are for $\Delta H^{\ddagger}/RT_R = 10$. Dashed lines are included for $\Delta H^{\ddagger}/RT_R = 5$ and 20 when for these values the function differed more than 2% from that for $\Delta H^{\ddagger}/RT_R = 10$.

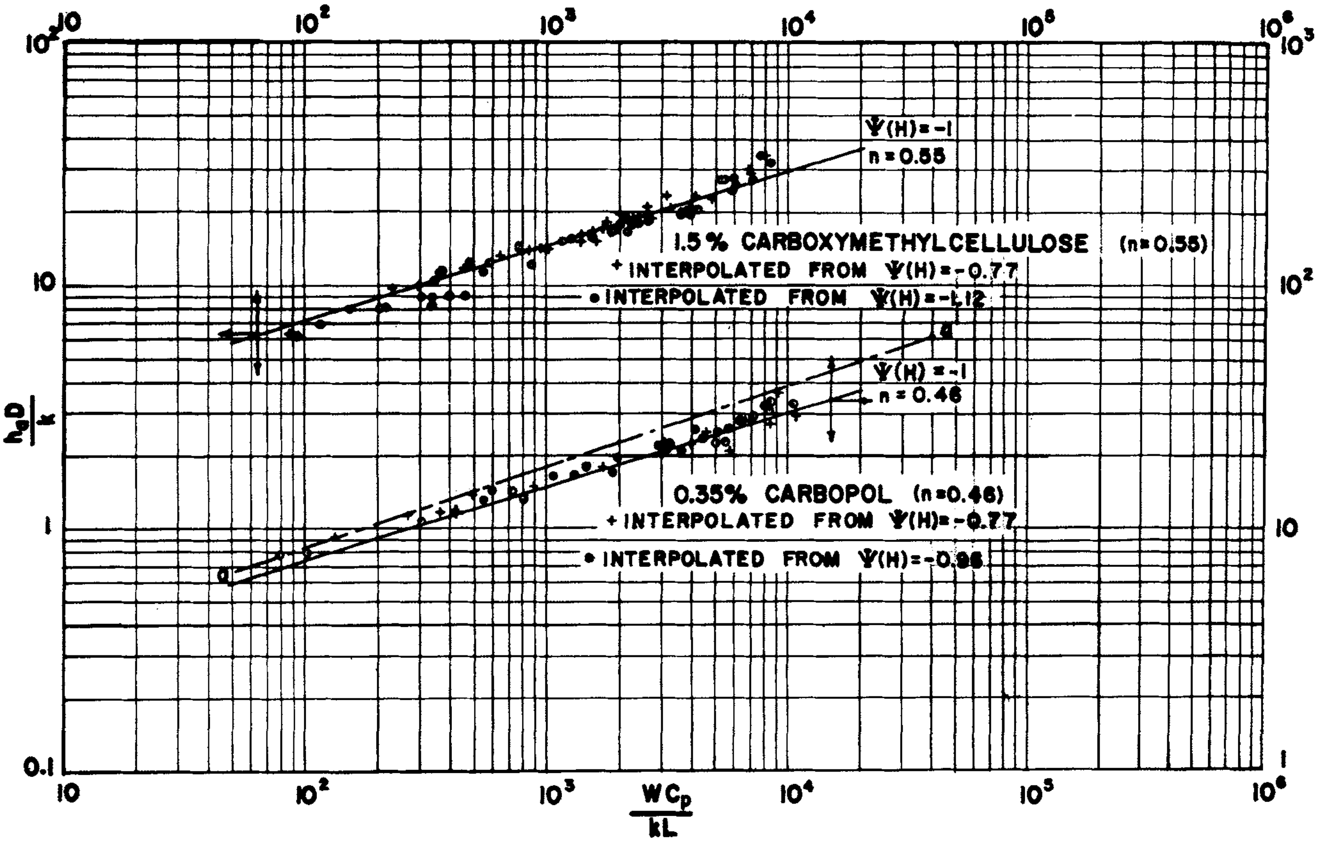

Fig. 5. Experimental data for aqueous 0.35% by weight CPM and 1.5% by weight CMC plotted together with the corresponding computed functions. The broken line a —a represents Equation (12) for the CPM data.

bopol) are probably no greater than the experimental error.

The data were taken in 1- and 4-ft test sections of ½-in. copper pipe (0.62 in. I.D. and 0.84 in. O.D.) preceded by calming sections 120 and 74 in. long respectively. The test sections were jacketed in 1¼-in. schedule 40 galvanized iron pipe with inlet and outlet cooling water tees alternating at 6-in. intervals but on opposite sides of the jacket. With this jacket arrangement and the high cooling-water flow rates, the pipe-wall temperature was constant throughout the jacket. The temperatures of the pipe wall were measured with calibrated copper-constantan thermocouples embedded with the aid of lead-tin solder in the copper pipe wall and located at 6-in. intervals along the pipe but alternately on one side and then the other. Inlet and outlet fluid temperatures were measured with calibrated precision tenth-degree thermometers located in a mixer at the outlet from the test section and in the inlet to the calming section but after the circulation pump. The thermocouples and thermometers were calibrated with each other by pumping the test fluid through both the copper pipe and the jacket. The flow properties of the fluids as a function of temperature were measured in a rotational viscometer (*11, 27*), which was calibrated with a standard oil. The constants for Equation (1) determined from the rotational viscometer data are tabulated in Table 1. The thermal conductivity, density, and heat capacity of the test fluids were taken to be the same as those of water. Previous studies (*6, 3*) showed that the thermal conductivity and density are essentially the same as for water. In these tests, L/D ratios were varied from 20 to 80, and temperature potentials to 40°C. were employed.

TABLE 1. CONSTANTS FOR EQUATION (1)

	m	n	$\Delta H^{\ddagger}$(cal./m)
1.6% CMC	0.32	0.55	5,600
0.35% CPM	2.1	0.46	4,720

Also plotted in Figure 5 as line *a-a* is Equation (12):

$$\frac{h_a D}{k} = 1.75\,\delta^{1/3}\left(\frac{Wc_p}{kL}\right)^{1/3}\left(\frac{\gamma_b}{\gamma_R}\right)^{0.14} \qquad (12)$$

for $n = 0.46$ and including the correction for the variation of consistency with temperature for the experimental data adjusted to $\psi(H) = 1.0$. This equation was suggested by Pigford (*24*) for the description of power law fluids and modified by Metzner et al. (*21*) to include the Sieder–Tate type correction $(\gamma_b/\gamma_R)^{0.14}$ for the variation of consistency with temperature.

As previously demonstrated for the case of heating (*9*), Equation (12) deviates significantly from the experimental data for cooling. Whereas it has been suggested (*22*) that Equation (12) overcorrects for the dependency of consistency on temperature, it appears that in the present case the correction is too small. Equation (12) without the correction for the dependency of consistency on temperatures as given by Pigford (*24*) fits the computations for $\psi(H) = 0$ very well (within 3%) at $N_{Gz} > 100$. Also it is clear that N_{Nu} may be much less than predicted by Equation (13), which was proposed in reference *21* for the case of highly dilatant fluids ($n \to \infty$),

$$\frac{h_a D}{k} = 1.59\left(\frac{Wc_p}{kL}\right)^{1/3} \qquad (13)$$

CONCLUSIONS

A numerical solution of the equations of motion and energy has been achieved for flow of non-Newtonian fluids in tubes of circular cross section with heat transfer to or from the fluid under the following conditions:

1. The flow properties are represented as a function of temperature by the Ostwald-deWaele power law [Equation (1)] with n, m, and $\Delta H^\ddagger$ independent of temperature.
2. The flow is steady state and rectilinear and is fully developed at the point where heat transfer begins.
3. The density, heat capacity, and thermal conductivity are independent of temperature and pressure.
4. Thermal energy generation in the fluid by viscous dissipation or any reaction is negligible.
5. Axial conduction of energy and momentum is negligible.
6. The temperature of the tube wall is constant.

The solutions which are represented in graphical form in Figures 1 to 4 have been previously shown (9) to be in excellent agreement with previous analytical solutions for fluids whose properties do not vary with temperature and to be in satisfactory agreement with extensive heat transfer data for the heating of 3 and 0.75% CMC solutions in water in horizontal 1-, 1½-, and 2-in. pipe for L/D ratios of 6 to 230, temperature increases up to 30°C., and temperature potential to 70°C. In the presently reported research the solutions have been shown to be in good agreement with experimental data for the cooling of 1.5 and 0.35% CPM solutions in water in a horizontal 0.62-in. I.D. tube, for L/D ratios of 20 to 80, and for temperature potentials to 40°C.

There is evidence that the solutions are exact and that they can be used with confidence in making heat transfer predictions for the many practical cases which are within the prescribed conditions. The predictions will be conservative if natural convection occurs.

ACKNOWLEDGMENT

The numerical program constitutes a part of a Ph.D. thesis by Gordon E. Jensen, and the experimental data are from an M.S. thesis by Fan-Sheng Tao. The assistance of Dean Edwin Dallin, a graduate student in the University of Utah Chemical Engineering Department, of the University of Utah Computer Center, and of the Western Data Processing Center in making the computations is gratefully acknowledged. Financial support was provided by the National Science Foundation.

NOTATION

A = constant in Equation (2), sec.$^{-1}$
B = constant in Equation (2), sq. cm./dyne
C_p = heat capacity, cal. g.$^{-1}$ °K.$^{-1}$
D = pipe diameter, cm.
e = base of natural logarithms, dimensionless
h_a = heat transfer coefficient based on average temperature difference, $h_a = \dfrac{q}{2\pi R_R L (T_R - T_a)}$, cal. cm.$^{-2}$ sec.$^{-1}$ °K.$^{-1}$
I_1 = $\displaystyle\int_{r^+}^{1} (r^+)^{1/n} \exp(-\Delta H^\ddagger/RT)\, dr^+$, dimensionless
I_2 = $\displaystyle\int_{0}^{1} I_1\, r^+\, dr^+$, dimensionless
k = thermal conductivity, cal. cm.$^{-1}$ sec.$^{-1}$ °K.$^{-1}$
L = length of heated tube, cm.
m = constant in Equation (1), dynes (sec.)^n (cm.)$^{-2}$
n = constant in Equation (1), dimensionless
N_{Br} = Brinkman number, dimensionless
N_{Gr} = Grashof number, dimensionless
N_{Gz} = Graetz number, (wC_p/kL), where C_p and k are evaluated at the average film temperature, $T_R/2 + T_i/4 + T_o/4$, dimensionless
N_{Nu} = Nusselt number, $(h_a D/k)$, where k is evaluated at the average film temperature, $T_R/2 + T_i/4 + T_o/4$, dimensionless
N_{Pr} = Prandtl number, dimensionless
N_{Re} = Reynolds number, dimensionless
P = pressure, dynes/sq.cm.
R = Boltzman constant (cal./mole/°K.) or pipe radius, cm.
r^+ = reduced radius (r/R), dimensionless
r = local radius, cm.
$\dot{S}$ = strain rate, $-du/dx$ ($-du/dr$ for tube flow), sec.$^{-1}$
T = absolute temperature, °K.
T_a = average bulk fluid temperature, $(T_i + T_o)/2$
T^+ = reduced temperature $(T - T_i)/(T_R - T_i)$, dimensionless
u = local velocity, cm./sec.
w = mass flow rate, g./sec.
z = axial coordinate, cm.
z^+ = N_{Gz}^{-1}

Greek Letters

α = diffusivity of thermal energy, $k/C_p\rho$, sq. cm./sec.
δ = $(3n + 1)/4n$, dimensionless
$\Delta H^\ddagger/R$ = constant in Equations (1) and (2), °K. [$\Delta H^\ddagger$ is interpreted (25) to be the activation energy for flow]
η = consistency, g. cm.$^{-1}$ sec.$^{-1}$
μ = constant in Equation (2), g. cm.$^{-1}$ sec.$^{-1}$
π = 3.1416
ρ = density, g.$_m$/cc.
τ = shear stress, dyne/sq. cm.
$\psi(H)$ = $(\Delta H^\ddagger/R)\ (1/T_i - 1/T_R)$, dimensionless

Subscripts

i = inlet
o = outlet
R = wall

LITERATURE CITED

1. Acrivos, A., *A.I.Ch.E. J.*, **4**, 285-289 (1958).
2. Andrade, E. N. da C., *Endeavour*, **13**, 117 (1954).
3. Beckstead, M. W., and J. M. Peterson, B.S. thesis, Univ. Utah, Salt Lake City (1961).
4. Beek, W. J., and R. Eggink, *Ingenieur*, **74**, 81-89 (1962).
5. Bird, R. B., *Chem. Engr. Tech.*, **31**, 569 (1959).
6. Boggs, J. H., and W. L. Sibbitt, *Ind. Engr. Chem.*, **47**, 2, 289-293 (1955).
7. Brinkman, H. C., *Appl. Sci. Res.*, **42**, 120-124 (1951).
8. Christiansen, E. B., *Chem. Eng. Progr.*, **60**, 8, 49-52 (1964).
9. ———, and S. E. Craig, Jr., *A.I.Ch.E. J.*, **8**, 154 (1962).
10. Christiansen, E. B., and Gordon E. Jensen, "Progress in International Research on Thermodynamic and Transport Properties," Academic Press, New York, pp. 738-747 (1962).
11. Christiansen, E. B., N. W. Ryan, and W. E. Stevens, *A.I.Ch.E. J.*, **1**, 544-548 (1955).
12. Craig, S. E., Jr., Ph.D. thesis, Univ. Utah, Salt Lake City (1959).
13. Graetz, L., *Ann. Phys.*, **18**, 79 (1883).
14. *Ibid.*, **25**, 337 (1885).
15. Grigull, U., *Chem. Ingr. Tech.*, **28**, 553, 655 (1956).
16. Harrison, William B., *ORNL-915* (June 25, 1954).
17. Jensen, Gordon E., Ph.D. thesis, Univ. Utah, Salt Lake City (1963).
18. Lemmon, Hal E., Ph.D. thesis, Univ. Utah, Salt Lake City (1963).
19. Lyche, B. C., and R. B. Bird, *Chem. Eng. Sci.*, **6**, 35 (1957).

20. Metzner, A. B., "Advances in Heat Transfer," J. P. Hartnet and T. F. Irvine, Jr., ed., Academic Press, New York, pp. 357-374 (1965).
21. ———, R. D. Vaughn, and G. L. Houghton, *A.I.Ch.E. J.*, **3**, 92-100 (1957).
22. Oliver, D. R., and V. G. Jenson, *Chem. Eng. Sci.*, **19**, 115-129 (1964).
23. Peterson, A. W., Ph.D. thesis, Univ. Utah, Salt Lake City (1960).
24. Pigford, R. L., *Chem. Eng. Progr. Symp. Ser. No. 17*, **51**, 79 (1955).
25. Ree, T., and H. Eyring, *J. Appl. Phys.*, **25**, 793-809 (1955).
26. Reid, Robert C., and Thomas K. Sherwood, "Chemical Engineering Series," McGraw-Hill, New York (1958).
27. Salt, D. L., N. W. Ryan, and E. B. Christiansen, *J. Colloid. Sci.*, **6**, 2, 146-154 (1951).
28. Schenk, I., and J. Van Laar, *Appl. Sci. Res.*, **A7**, 449 (1958).
29. Schneider, P. J., *Trans. Am. Soc. Mech. Engrs.*, **79**, 765-773 (1957).
30. Tao, Fan-Sheng, M.S. thesis, Univ. Utah, Salt Lake City (1964).
31. Thomas, D. G., A.S.M.E. Second Symp. Thermophysical Properties, pp. 704-717, Academic Press, New York (1962).

Manuscript received April 4, 1966; paper accepted August 15, 1966. Paper presented at A.I.Ch.E. Detroit meeting.

The Effect of Recycle on a Linear Reactor

W. R. SCHMEAL and NEAL R. AMUNDSON

University of Minnesota, Minneapolis, Minnesota

The theoretical transient behavior of an isothermal packed bed or tubular reactor with direct recycle is investigated. It is shown that the recycle effect, coupled with the phenomenon of axial dispersion, causes waves in the reactant concentration to travel through the bed when the feed concentration undergoes a step change. The waves have a length almost equal to the bed length and they travel with the velocity of the fluid. The behavior of the waves is very insensitive to the value of the Peclet number.

The pertinent linear differential equation is solved by the method of generalized Fourier transforms and the boundary conditions are such that the Sturm-Liouville theorem cannot be used. The operator is nonself-adjoint and the eigenfunctions are not mutually orthogonal. The eigenvalues, which are complex, are found by means of the argument principle. Sample calculations are presented of the first one hundred terms of the Fourier expansion of a solution function and this is compared to a simplified approximate series which is developed.

Recycle streams are often employed with packed beds or tubular reactors for temperature control, inhibition of undesired side reactions, or efficient use of reactants. Recycling effects feed back of information from the reactor exit to the entrance. This may couple with the phenomenon of axial dispersion that provides communication throughout the fluid, and aggravate or reenforce instabilities.

How a step change in feed concentration of reactant effects damped oscillations in the concentration profile within the bed is shown here for the relatively simple case of an isothermal reactor with a first-order reaction occurring. For mathematical simplicity the case of direct recycle is considered, the method being amenable to the inclusion of a linear process or time lag in the recycle stream. In actual practice, fractionation and heat exchange processes may be present in the recycle stream which are complicated to analyze and have a strong effect on the stability of the system. The example chosen should however give qualitative information about the dynamics of other processes which incorporate recycle, such as polymerization reactors and crystallizers (*8*).

The law of conservation of mass applied to the isothermal flow reactor produces a linear differential equation for the concentration of reactant c as a function of time t and of axial position x.

$$D\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} - kc = \frac{\partial c}{\partial t} \qquad (1)$$

The term on the left represents a diffusive flux of reactant, the second a convective flux, and the last a volumetric rate of depletion via reaction, while the term on the right accounts for the local accumulation rate.

The form of the equation is

$$L(c) = \partial c/\partial t \qquad (2)$$

W. R. Schmeal is with Shell Development Company, Emeryville, California.